

THE LINEAR TRACE HARNACK QUADRATIC ON A STEADY GRADIENT RICCI SOLITON SATISFIES THE HEAT EQUATION

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All quantities shall be assumed C^∞ .

1. Evolution of the linear trace Harnack on steady and shrinking gradient Ricci solitons

Suppose $(\mathcal{M}^n, g_{ij}(t), h_{ij}(t))$ satisfies the linearized Ricci flow $\frac{\partial}{\partial t} g_{ij} = -2R_{ij}$ and $\frac{\partial}{\partial t} h_{ij} = (\Delta_L h)_{ij}$, where Δ_L denotes the Lichnerowicz Laplacian. Define the matrix Harnack quantities (see Hamilton [3])

$$M_{pq} = \Delta R_{pq} - \frac{1}{2} \nabla_p \nabla_q R + 2R_{pijq} R^{ij} - R_{pk} R_q^k, \quad P_{ipq} = \nabla_i R_{pq} - \nabla_p R_{qi}.$$

Recall that if $X(t)$ is any time-dependent vector field on \mathcal{M} , then the corresponding linear trace Harnack quantity $Z(h, X) = \text{div}(\text{div}(h)) + \langle \text{Rc}, h \rangle + 2 \langle \text{div}(h), X \rangle + h(X, X)$ satisfies (see [2])

$$(1) \quad \left(\frac{\partial}{\partial t} - \Delta \right) Z = 2h^{pq} (M_{pq} + 2P_{ipq} X^i + R_{pijq} X^i X^j) \\ - 4 (\nabla_j X^i - R_j^i) \nabla^j (\text{div}(h)_i + h_{ik} X^k) \\ + 2 (\text{div}(h)_j + h_{ij} X^i) \left(\frac{\partial X^j}{\partial t} - \Delta X^j - R_k^j X^k \right) \\ + 2h_{ij} (\nabla_p X^i - R_p^i) (\nabla^p X^j - R^{pj}).$$

Since $\frac{\partial}{\partial t} \text{Rc} = \Delta_L \text{Rc}$, we may take $h = \text{Rc}$. Then $2Z(\text{Rc}, X) = \Delta R + 2|\text{Rc}|^2 + 2 \langle \nabla R, X \rangle + 2 \text{Rc}(X, X)$, which is Hamilton's trace Harnack quadratic. Furthermore, if $\text{Rc} + \nabla \nabla f = 0$, then $Z(\text{Rc}, -\nabla f) = 0$ by $\Delta R + 2|\text{Rc}|^2 = \langle \nabla R, \nabla f \rangle$ and $2R_{ij} \nabla_j f = \nabla_i R$.

Now consider the linear trace Harnack quantity for the linearized Ricci flow on a steady gradient Ricci soliton.

Lemma 1. *If $(\mathcal{M}^n, g(t), f(t), h(t))$ satisfies $\frac{\partial}{\partial t} g = -2\text{Rc} = 2\nabla \nabla f$, $\frac{\partial f}{\partial t} = \Delta f$, $\frac{\partial}{\partial t} h = \Delta_L h$, then $Z(h, -\nabla f) = \text{div}(\text{div}(h)) + \langle \text{Rc}, h \rangle - 2 \text{div}(h)(\nabla f) + h(\nabla f, \nabla f)$ satisfies $\frac{\partial Z}{\partial t} = \Delta Z$.² I.e., the linear trace Harnack quantity solves the heat equation.*

Proof. By $M_{pq} = P_{ipq} \nabla^i f$, $P_{ipq} = R_{pijq} \nabla^j f$, $\nabla_j \nabla^i f + R_j^i = 0$, and $((\frac{\partial}{\partial t} - \Delta) \nabla f)^j = R_k^j \nabla^k f$, for $Z(h, -\nabla f)$ each of the terms on the RHS of (1) (with $X = -\nabla f$) are zero. \square

Remark. (1) The quantity $Z(h, -\nabla f)$ arises naturally from the following consideration. If $g_{ij}(s), f(s)$ are such that $\frac{\partial f}{\partial s} g_{ij} = h_{ij}$ and $\frac{\partial f}{\partial s} = \frac{H}{2}$, where

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²The same is true if we replace $\frac{\partial f}{\partial t} = \Delta f$ by $\frac{\partial f}{\partial t} = |\nabla f|^2$ essentially since $R = -\Delta f = 1 - |\nabla f|^2$.

$H = g^{ij}h_{ij}$ (so that $\frac{\partial}{\partial s}(e^{-f}d\mu) = 0$; see [6]), then Perelman's scalar curvature satisfies

$$\frac{\partial}{\partial s} \left(R + 2\Delta f - |\nabla f|^2 \right) = Z(h, -\nabla f) - 2 \langle h, \text{Rc} + \nabla \nabla f \rangle.$$

Note for a steady gradient Ricci soliton, $\frac{\partial}{\partial s} \left(R + 2\Delta f - |\nabla f|^2 \right) = Z(h, -\nabla f)$ while $\frac{\partial}{\partial t}(e^{-f}d\mu) = (-\Delta f - R)e^{-f}d\mu = 0$.

(2) If $g(t), f(t)$ solves $\frac{\partial g}{\partial t} = -2\text{Rc}$ and $\frac{\partial f}{\partial t} = -\Delta f + |\nabla f|^2 - R$, then $V \doteq (2\Delta f - |\nabla f|^2 + R)e^{-f}$ satisfies $\square^* V = -2|R_{ij} + \nabla_i \nabla_j f|^2 e^{-f}$, where $\square^* = -\frac{\partial}{\partial t} - \Delta + R$ (see [6]).

The shrinker analogue of Lemma 1 is the following.

Lemma 2. *If $(\mathcal{M}^n, g(t), f(t))$, $t < 0$, satisfies $\frac{\partial}{\partial t}g = -2\text{Rc} = 2\nabla \nabla f + \frac{1}{t}g$, $\frac{\partial f}{\partial t} = |\nabla f|^2 = \Delta f - \frac{1}{t}f$, and $\frac{\partial}{\partial t}h = \Delta_L h$, then $(\frac{\partial}{\partial t} - \Delta)(t^2(Z(h, -\nabla f) + \frac{H}{2t})) = 0$.*

Proof. Applying $M_{pq} + \frac{1}{2t}R_{pq} = P_{ipq}\nabla^i f$, $P_{ipq} = R_{pijq}\nabla^j f$, $R_{ij} + \nabla_i \nabla_j f = -\frac{1}{2t}g_{ij}$, and $((\frac{\partial}{\partial t} - \Delta)\nabla f)^j - R_k^j \nabla^k f = -\frac{1}{t}\nabla^j f$ to (1) yields $(\frac{\partial}{\partial t} - \Delta)Z(h, -\nabla f) = -\frac{2}{t}Z(h, -\nabla f) - \frac{1}{t}\langle h, \text{Rc} \rangle - \frac{H}{2t^2}$. Because $(\frac{\partial}{\partial t} - \Delta)H = 2\langle h, \text{Rc} \rangle$, we conclude

$$\left(\frac{\partial}{\partial t} - \Delta \right) \left(Z(h, -\nabla f) + \frac{H}{2t} \right) = -\frac{2}{t} \left(Z(h, -\nabla f) + \frac{H}{2t} \right).$$

□

Remark. If $h = \text{Rc}$, then for a shrinker, since $\frac{1}{2}\Delta R + |\text{Rc}|^2 = \frac{1}{2}\langle \nabla R, \nabla f \rangle - \frac{R}{2t}$ and $H = R$, we have that $Z(h, -\nabla f) + \frac{H}{2t} = Z(\text{Rc}, -\nabla f) + \frac{R}{2t} = 0$.

2. Interpolating between Perelman's and Cao-Hamilton's Harnacks on a steady soliton

Now consider the system $\frac{\partial}{\partial t}g_{ij} = -2R_{ij} = 2\nabla_i \nabla_j f$, $\frac{\partial f}{\partial t} = |\nabla f|^2$, and $\frac{\partial u}{\partial t} = \Delta u + Ru$, $u > 0$ (when $n = 2$ this is the linearized Ricci flow).³ Mimicking Li, Yau, and Hamilton, define $v = \log u$, $Q = \Delta v + R$, and $L = \frac{1}{2}(\frac{\partial}{\partial t} - \Delta) - \nabla v \cdot \nabla$. Then

$$(2) \quad LQ = |\nabla \nabla v|^2 + \langle \text{Rc}, \nabla \nabla v \rangle + \text{Rc}(\nabla(v - f), \nabla(v - f)).$$

Following X. Cao and Hamilton [1], define the Harnack $P = 2Q + |\nabla v|^2 + R$. Since

$$L(|\nabla v|^2 + R) = |\text{Rc}|^2 - |\nabla \nabla v|^2,$$

we have that $P = 2\Delta v + |\nabla v|^2 + 3R$ satisfies the parabolic Bochner-type formula⁴

$$LP = |\nabla \nabla v + \text{Rc}|^2 + 2\text{Rc}(\nabla(v - f), \nabla(v - f)).$$

In particular, if $\text{Rc} \geq 0$, then $LP \geq \frac{1}{n}Q^2 \geq 0$.

More generally, if $\frac{\partial u}{\partial t} = \varepsilon^{-1}\Delta u + Ru$ (interpolating), where $\varepsilon \in \mathbb{R} - \{0\}$, then $P_\varepsilon = 2\Delta v + |\nabla v|^2 + (2\varepsilon + 1)R$ satisfies the heat equation

$$\begin{aligned} L_\varepsilon P_\varepsilon &= \varepsilon^{-1} |\nabla \nabla v|^2 + 2 \langle \text{Rc}, \nabla \nabla v \rangle + \varepsilon^{-1} |\text{Rc}|^2 \\ &\quad + 2\varepsilon^{-1} \text{Rc}(\nabla(v - \varepsilon f), \nabla(v - \varepsilon f)) + (1 - \varepsilon^{-1}) \text{Rc}(\nabla(v + f), \nabla(v + f)), \end{aligned}$$

³Note that $\frac{\partial}{\partial t}e^f = \Delta e^f + R e^f$. The trivial case of the following calculations is $v = f$.

⁴Note that $\nabla \nabla v + \text{Rc} = \nabla \nabla(v - f)$.

where $L_\varepsilon = \frac{1}{2}(\frac{\partial}{\partial t} - \varepsilon^{-1}\Delta) - \varepsilon^{-1}\nabla v \cdot \nabla$. The Ricci terms may also be rewritten as $(1 + \varepsilon^{-1}) \operatorname{Rc}(\nabla(v - f), \nabla(v - f)) + 2(\varepsilon - \varepsilon^{-1}) \operatorname{Rc}(\nabla f, \nabla f)$. If $\varepsilon = -1$, we have

$$\left(\frac{1}{2}\left(\frac{\partial}{\partial t} + \Delta\right) + \nabla v \cdot \nabla\right) \left(2\Delta v + |\nabla v|^2 - R\right) = -|\operatorname{Rc} - \nabla\nabla v|^2 = -|\nabla\nabla(f + v)|^2.$$

This is a special case of Perelman's pointwise energy monotonicity formula. We interpolated the calculations but not the estimates between $\varepsilon^{-1} = -1$ and $\varepsilon^{-1} = 1$; for successful interpolations between Li–Yau–Hamilton inequalities, see Ni [4], [5].

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